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Book Review of Paul Cobb (ed.) Learning Mathematics: Constructivist and interactionist theories of mathematical development

The book is a reprint of a special issue of the journal *Educational Studies in Mathematics* (Vol.26, 2–3, 1994), for which Paul Cobb acted as guest editor. It contains the editor's preface and six articles by different authors, each of which deals with a different aspect of mathematics learning. This variety makes for interesting reading, because the individual topics are approached from remarkably similar points of view. The six authors all adhere to the basic constructivist principle that learning is not a passive receiving of ready-made knowledge but a process of construction in which the students themselves have to be the primary actors. But they also show in the accounts of their teaching experiments that this principle in no way precludes the notion that the students' individual constructing is constantly stimulated, constrained, and thus directed by interactions with each other and with a teacher.

L. P. Steffe & H. G. Wiegel's piece "Cognitive play and mathematical learning in computer microworlds" is the first of the sequence and it begins with the question: "Is there a model of mathematical learning powerful enough to be useful to mathematics teachers in their teaching?" (p.7). They describe "Toys" and "Sticks," two sophisticated computerized microworlds which offer, as they say, "a starting point to transform children's cognitive play activity into a mathematical play activity" (p.11). They are dealing with elementary school children and the mathematics consists of the basic operations of arithmetic.

Two things emerge clearly from the protocols they report. First, the introduction of a well-designed and easily accessible microworld does indeed generate the students' motivation to play with and in it. Second, it is not at all difficult for a sensitive teacher to steer the students' play towards mathematically relevant aspects and thus to turn it into a *mathematical* activity that has playful character.

At the end of their exposition the two authors can therefore answer their initial question and claim on rather solid grounds that the method of using the microworlds they have designed does indeed constitute a model of mathematical learning, and they justify this claim by focusing on their system's capacity to generate motivation:

> … mathematics is a product of the functioning mind … children will not sustain a mathematical activity unless they experience satisfaction in the process of that activity. This satisfaction can emerge from the use of their already available schemes …in a dynamic learning environment, from a successful organization of their experience, and from playful orientation of the activity (p.28).

J. Confrey & E. Smith start the second paper of the book by reminding the reader that the errors students make “are seldom random or capricious – they have a rationality and functionality of their own” (p.31). Hence, whatever a student does or says in the context of a mathematical problem must be taken as a *possible* indication of that student’s present network of mathematical concepts and relations.

Working in the area of multiplicative relations and functions, the authors’ minute analyses of students’ actions and reactions show that the concept of “rate of change” is accessible to students long before they are faced with calculus in school. Hence, a “covariation” approach to functional relationships would seem far more promising than the “correspondence” approach traditionally employed.

The paper also provides a welcome further exposition of Confrey’s original notion of “splitting,” a novel approach to the understanding of multiplicative relations This approach is, I believe, a particularly helpful one because it establishes a link between an abstract operation and a common sensorimotor activity.

From the perspective of my own work on the abstraction of the concept of number (Glasersfeld 1987), I would not agree with Confrey & Smith’s analysis of the construction of *units*. They compound the mental construction of unit and plurality by ”broadening” the *definition* of unit “to include other underlying repeated actions” (p.38). This, I think, is not a useful simplification because it confuses two separate conceptual steps. Operational awareness of carrying out a repetition is indispensable in generating pluralities, but it is not a requisite for the conceptual construction of units. Discrete individual units have to be available in order to serve as the material for pluralities. Subsequently, of course, a plurality can be unitized to form a new composite unit – but this pertains to the *use* of the unitizing operation, not to its definition. This confusion could have been avoided if the authors had from the very beginning discriminated between the generic concept of unit and the iterative use of an already constructed unit as “standard” or ”measure” in a given situation.

This minor technical criticism does not diminish the value of the ”multi-conceptual approach to rate” which the authors are proposing, and their suggestion that “it holds particular promise in preparing students for calculus” (p.57) is likely to be right.

People who have to teach calculus or are simply interested in its conceptual underpinnings, will be fascinated by the way P. W. Thompson develops the approach to this domain in the fifth section of the book.

He, too, sets out by stressing the role of sensorimotor experience in the formation of mathematical concepts and reminds us of Piaget’s eminently useful classification of mental images. In the course of this, he provides a welcome review of various authors’ use of the term “mental image.” Then, with the help of partial descriptions of several teaching experiments, he shows among other things how important it is that students “come to interpret notation as someone’s attempt to say something” (p.138).

This, it seems to me, is a crucial point. The vapid slogan that mathematics is a sophisticated form of symbol manipulation leads too many teachers and students to forget that a mark on paper or a speech sound cannot be understood as a symbol, unless it is associated with a conceptual structure in the user’s head. With the help of protocols and his own observations of the students’ behavior, Thompson shows just how difficult it is to complement the formal steps of a procedure involving functions of infinitesimals with mental images suitable for a proper understanding of calculus. Although his exposition is still far from providing a primer of images that could be used didactically, he provides sufficient examples to point in this fruitful direction. Given the helplessness that many teachers experience in this particular area of instruction, such examples would seem to be an extremely valuable contribution.

In "The gains and the pitfalls of reification – The case of algebra," the book's fourth section, A. Sfard & L. Linchevski launch their exposition by developing the distinction between *operational* and *structural*. They illustrate for the reader the widely accepted view that the two terms indicate "ostensibly incompatible ways of seeing mathematical constructs" (p.89) On the following page, they contrast this view with their own, which they summarize by saying: "… mathematical objects are an outcome of *reification* – of our mind's eye's ability to envision the result of processes as permanent entities in their own right."

In this connection, it is worth drawing attention to the fact that the ambiguous reference to either process or result is a phenomenon that is not peculiar to algebra or mathematics. It is endemic in our natural languages. There are hundreds of words, such as "walk" or "construction," that can mean either an activity or the activity's result. In all these cases, the reference to the result does not become comprehensible unless one is able to re-present to oneself the activity that produced it. Indeed, conceptual analysis shows that the result contains the activity that has been turned into an entity by the *unitizing* operation that was mentioned by Confrey & Smith in Chapter 2. Our languages show that we have the conceptual ability to image an activity in time, and to frame this ongoing process as a unitary item. That we can do this, is perhaps best illustrated in the practice of operational definitions (Bridgman, 1936) where static, thing-like concepts are characterized by the physical or mental operations that produce them as result. This ability is, of course, one of the basic building blocks in mathematics, from the abstraction of numerosity from a count (e.g. when we say "One, two, three – there are three") to the use of infinite progressions as units in further operations.

As Sfard & Linchevski say: "…a closer look at these [reified] entities will reveal that they cannot be separated from the processes themselves as self-sustained beings" (p.90). It is therefore a little misleading to use Bohr's notion of "complementarity" (p.89) to lead up to the duality in question, because what is considered complementary in quantum physics are two explanatory models, both containing processes and their results, but mutually incompatible on the conceptual level.

Having established the difference between operational processes and the resulting "reified" structures, the authors present a most interesting historical survey of the development of this dichotomy and then proceed to demonstrate its application in teaching situations.

J. Voigt's paper, "Negotiation of mathematical meaning and learning mathematics," is the only one of the six that specifically focuses on social interaction. He shows his own approach in a variety of actual class episodes, and then gives a more or less standard characterization of the difference between the radical interpretation of Piaget's constructivism and some social interactionist views:

> On the one hand, the individual is the actor (subject), and the mathematical knowledge is constructed by the actor. On the other hand, the individual is the object of cultural practices, and given mathematical knowledge is internalized (p.187).

Constructivists would balk at the suggestion that given knowledge can be "internalized" and they would be inclined to ask how this is supposed to happen. But Voigt has already shown that his own approach is more sophisticated. As his title says, he focuses on the negotiation of meanings, and the first example he gives makes clear how he intends this. A picture from a textbook purports to represent a univocal arithmetic task, but the students to whom it is shown come up with five quite different spontaneous interpretations. Interactionists, Voigt comments, assume "that every object or event in human interaction is plurisemantic. … In order to make sense of that object or event, the subject uses his/her background knowledge and forms a sensible context for interpreting the object" (p.174). The negotiation, then, concerns the establishing of consensus about the desired interpretation. In other words, social interaction leads the participating individuals to adjust what they see and verbalize until there appears to be mutual compatibility. Wherever this adaptation is successful, an observer can, as Cobb says in his preface, speak of “taken-to-be-shared meanings” (p.4).

Viewed in this way, the apparent opposition of radical and social constructivism disappears and the two approaches can be seen as complementary. Voigt’s paper does much to foster this integration. It also brings out a number of interesting didactic problems. For instance:

> Students must learn to distinguish between mathematical arguments and those conventions of mathematizing which are established in order to make the negotiation of mathematical meaning easier in school. This is a difficult point, because mathematics itself is full of conventions invented to make the mathematical communication easier (p.191).

As an example he uses the addition of negative numbers. There is no question that placing a minus sign before a numeral is a convention invented by mathematicians in order to communicate about these mathematical “objects,” but a student’s understanding of the conventional symbol’s *meaning* requires something more than negotiation. The concept of negative number was generated by applying an abstraction made from counting with objects *to* a domain that extends beyond the world of objects. In his preface to the book, Cobb mentions that “reflective experiences that involve reviewing prior activity and anticipating the results of potential activity” (p.1) are an integral part of mathematical experience. This is a paraphrase of Piaget’s fundamental notion that mathematical concepts are always results of *reflective abstraction*. There is no immediate connection between such abstractions and communication. Indeed, their communication requires that they be associated with conventional symbols which, in order to be interpreted, require the presence of *a fitting* reflective abstractions in the head of the interpreter.

Voigt suggests some of this in the paragraphs that follow, but the distinction between students’ individual reflective abstractions and communication is subsumed by the term “negotiation.” There is nothing wrong with the term, but it tends to suggest that the necessary abstractions will happen automatically in the course of social interaction – and this, in my view, is a general weakness of the interactionist and social constructionist positions. There is no doubt that acquiring the proficient use of mathematical symbols, just as acquiring the use of language, requires social interaction, but this does not do away with the fact that in both cases *meaning* has to be abstracted by each prospective individual user in his or her own experiential field.

In this review I have left the paper by S. Pirie & T. Kieren, which comes third in the book, for the end, because, unlike the others, it presents a general model of learning. It is a sophisticated, multidimensional model of understanding that has occupied the two authors, for several years. Their presentation of in this book succeeds in making it transparent for the reader. This is achieved by interlacing the description with the story of Teresa, a twelve-year old’s learning experience in the area of fractions, as well as a couple of briefer episodes with other children.

The structure of the model is graphically represented by a diagram consisting of eight circles whose centers are equidistant on the horizontal radius of the largest. The succession of inscribed circles indicates qualitatively different levels of understanding. Each level is further divided into two areas, and the labels of these areas – e.g. “image doing” versus “image reviewing” – suggest that the division is not unlike the Piagetian division of the figurative and the operative, respectively of empirical versus reflective abstractions. The arrangement of the embedded circles indicates a hierarchy of conceptual activities but not a one-way street. Although conceptualization always begins in the smallest circle and proceeds outward, it can at any point “fold back” to previous levels. This enables the model to account for the frequent observation that a thinker faced with a problem in an unfamiliar context is likely to revert to forms of reasoning that are not as sophisticated as those developed in some other situations.

In my view, the Pirie/Kieren model is extremely useful because it provides a powerful instrument for seeing and analysing intellectual development. Its eight levels capture much of what Piaget's genetic epistemology suggested but did not actually specify. Here we are given a detailed account of different ways in which reflection and abstraction can operate, of how the formation of symbols leads to new ways of operating, and of the recurrent use of sensorimotor images in the construction of highly abstract concepts. To use a metaphor: this model is a microscope that allows us to see a fine-grained structure in the process of developing understanding.

To characterize the book as a whole, one can do no better than quote a few of the many felicitous formulations of the six authors.

Sfard & Linchevski state the fundamental maxim: "It seems very important that we try to motivate our students to actively struggle for meaning at every stage of the learning" (p.121).

Confrey & Smith remark that the researcher/teacher's conceptual network, the perspective from which he or she observes students and infers and categorizes their operations, can be modified in the course of a teaching experiment (p.32). Whenever this happens, the teachers' experience and their mathematical knowledge may be enriched. Voigt explicitly makes the same point (p 189), and the other contributions show it, even if they do not state it in so many words.

This last point is one that should make this book fruitful reading for teachers. The realization that their mathematical horizon can be expanded by ideas coming from the students, will tend to lead to two important insights. On the one hand, it will encourage the teacher to think of the students, not as passive receivers, but as active, autonomous thinkers. On the other, it will make palpable that knowledge of mathematics must not be considered a fixed, immutable body of facts, concepts, and methods, that must always be approached and understood in one and the same way.

In the community of didactic research, this book should do much to dispel the frequently voiced criticism that the constructivist orientation ignores the role of the teacher. All the contributions demonstrate that teachers are indispensable if students are to construct a network of mathematical concepts and operations adequate to the tasks they are likely to meet later in their lives. But all the contributions also make a strong case for the view that neither such a network nor its components can be transmitted to the students from the outside. The students themselves have to build them. How teachers may operate in view of this goal, is shown in the many protocols that enliven this text. Occasionally it is made explicit – as for instance when Pirie & Kieren say that teachers should provide students with opportunities to test their ideas in new situations. Such situations can be chosen so as to facilitate the students extension or modification of concepts in a desirable direction (p.77). The teacher, thus, is no longer seen as the supplier of knowledge but rather as the impresario who has the task of directing the students conceptual constructing.

I said at the outset of this review that the book makes interesting reading because it presents a variety of different topics and at the same time a number of unifying ideas in the area of mathematics education. I want to add that the presentation and discussion of the many protocols from teaching experiments are bound to fascinate readers regardless of the particular school of thought they might subscribe to. To witness the actual struggles and successes of elementary or secondary school children who are developing mathematical understanding is as thrilling as it is instructive. Paul Cobb has succeeded in putting together a collection of documents that many readers will want to keep on their desks as a permanent reference.

As postscript I should like to add that, since this book is likely to reach a second edition, a careful correction of printer's errors would be in order. There are quite a few, illustrating the fact that computerized spell-checkers do no more than check individual words – the decision whether a word fits the context, still requires a human reader.

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